A bipartite class of entanglement monotones for *N*-qubit pure states

2004 J. Phys. A: Math. Gen. 37 8293

(http://iopscience.iop.org/0305-4470/37/34/008)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 8293–8302 PII: S0305-4470(04)80527-4

A bipartite class of entanglement monotones for *N*-qubit pure states

Clive Emary

Instituut-Lorentz, Universiteit Leiden, PO Box 9506, 2300 RA Leiden, The Netherlands

Received 11 May 2004
Published 11 August 2004
Online at stacks.iop.org/JPhysA/37/8293
doi:10.1088/0305-4470/37/34/008

Abstract

We construct a class of algebraic invariants for N-qubit pure states based on bipartite decompositions of the system. We show that they are entanglement monotones, and that they differ from the well-known linear entropies of the subsystems. They therefore capture new information on the non-local properties of multipartite systems.

PACS numbers: 03.67.Mn, 03.65.Ud

1. Introduction

In contrast to bipartite systems, the nature of entanglement in multipartite systems is at present only partially understood. An important step forward would be the determination of a complete set of algebraic invariants (AI) for multipartite systems of arbitrary size. For an N-particle pure state, $|\Psi\rangle = \sum a_{b_1,\ldots,b_N}|b_1\rangle \otimes \cdots \otimes |b_N\rangle$, the AIs are algebraic functions of the coefficients $a_{b_1,\ldots,b_N}$ which are invariant under local unitary transformations (LU) [1].

Although invariance under LU is crucial, it is not the whole story, since one usually considers a more general situation in which the parties can perform additional non-unitary operations (such as local measurements), and can communicate classically with one another. The combinations of local operations and classical communication is denoted by LOCC. The pertinent measures of entanglement here are the entanglement monotones (EMs). These are AIs which are non-increasing, on average, under LOCC [2].

Whether speaking of the AIs or the EMs, the number required to entirely specify the non-local properties of the system increases exponentially with the number of particles [1], and thus a complete description seems out of reach for large systems. Indeed, a complete set of AIs are only known for systems of up to four qubits [3]. Consequently, in considering multiparticle entanglement, we must seek useful measures that capture essential features of the entanglement, and/or are simple to calculate.

Emerging as the most important EM for pure states is the hyperdeterminant Δ [4]. The hyperdeterminant is afforded this status because it is nonzero only for those states which

possess genuine N-particle entanglement. For two qubits, Δ is the concurrence [5] and for three qubits, the tangle [6]. For systems of more than four qubits, the explicit calculation of Δ is highly nontrivial.

Other useful EMs exist for N-qubit systems. A good example is the von Neumann entropy (or its linearized form) of a single qubit with the rest of the system. This tells us whether the qubit in question is separable or not. Meyer and Wallach (MW) [7] introduced an N-qubit entanglement measure which, as Brennen has shown [8], is equivalent to the average of all the single-qubit linear entropies.

MW construct these linear entropies in a particularly elegant fashion, and in this article we introduce a new family of EMs obtained from a generalization of this construction. We show that these quantities are EMs, and that they reflect an aspect of the entanglement different to that captured by the linear entropies of all sub-systems. The utility of these EMs is demonstrated by considering the four qubit system. Here, our EMs reproduce the fundamental algebraic invariants recently described by Luque *et al* [3], and also prove useful in differentiating between the nine families of four-qubit entangled states recently described by Verstraete *et al* [9]. The simplicity of our construction gives the prospect of extending these results to larger numbers of qubits.

The paper proceeds as follows. In section 2 we describe the construction of these entanglement monotones. In section 3, we consider some important properties; we demonstrate that they are indeed EMs and compare them with linear entropies. We consider in detail the four qubit system in section 4, and conclude with a discussion in section 5. In the appendix we give the details of the proofs used here.

2. Construction

A pure state of N qubits can be written as

$$|\Psi\rangle = \sum_{b_1,\ldots,b_N} a_{b_1,\ldots,b_N}|b_1\ldots,b_N\rangle = \sum_{X=0}^{L-1} a_X|X\rangle \tag{1}$$

where X is the decimal for the binary string $b_1,\ldots,b_N$ such that $0 \leqslant X \leqslant L-1$ with $L = 2^N$. Meyer and Wallach [7] introduced the single-qubit 'reduction operators' $\iota_B^{(k)}$, which act on qubit k in the following manner:

$$\iota_B^{(k)}|b_1,\ldots,b_N\rangle = \delta_{b_k,B}|b_1,\ldots,\hat{b}_k,\ldots,b_N\rangle. \tag{2}$$

The circumflex denotes absence. As $B \in \{0,1\}$ can take on one of two values, the action of reduction operators at locus k of $|\Psi\rangle$ gives the two vectors

$$\iota_0^{(k)}|\Psi\rangle = \left|\mathbf{V}_0^{(k)}\right\rangle = \sum_{X=0}^{L/2-1} V_0^{(k)}(X)|X\rangle \tag{3}$$

$$\iota_1^{(k)}|\Psi\rangle = \left|\mathbf{V}_1^{(k)}\right\rangle = \sum_{X=0}^{L/2-1} V_1^{(k)}(X)|X\rangle \tag{4}$$

which MW combine to form the N quantities

$$D_1^{(k)} = 4\sum_{X<Y} |V_0(X)V_1(Y) - V_0(Y)V_1(X)|^2. \tag{5}$$

They then define their entanglement measure $\mathcal{Q}_1$ as the average over all qubits k of these quantities

$$\mathcal{Q}_1 \equiv \frac{1}{N}\sum_{k=1}^{N} D_1^{(k)}. \tag{6}$$

As we see below, the quantities $D_1^{(k)}$ are themselves EMs, and are equal to the linear entropies of the kth qubit [8].

We extend the MW construction by introducing reduction operators $I^{(k_1,\ldots,k_n)}_{B_{k_1},\ldots,B_{k_n}}$ that act on n qubits,

$$I^{(k_1,\ldots,k_n)}_{B_{k_1},\ldots,B_{k_n}}|b_1,\ldots,b_N\rangle \equiv \delta_{b_1,B_{k_1}},\ldots,\delta_{b_n,B_{k_n}}|b_1,\ldots,\hat{b}_{k_1},\ldots,\hat{b}_{k_n}\ldots b_N\rangle. \tag{7}$$

We reference these operators by the locus $\{k\} \equiv k_1,\ldots,k_n$ describing the qubits on which I acts (the reduced qubits), and by the decimal X corresponding to the bit-string $B_{k_1},\ldots,B_{k_n}$. The integer n runs from unity to either $N/2$ or $(N-1)/2$ depending on whether N is even or odd. The action of $I_X^{\{k\}}$ on $|\Psi\rangle$ is to produce the $(N-n)$-qubit state

$$I_X^{(k_1,\ldots,k_n)}|\Psi\rangle = \left|\mathbf{V}_X^{(k_1,\ldots,k_n)}\right\rangle = \sum_{Y=0}^{\bar{L}-1} V_X^{(k_1,\ldots,k_n)}(Y)|Y\rangle \tag{8}$$

with $\bar{L} \equiv 2^{N-n}$. For a given locus $\{k\}$, there are $l \equiv 2^n$ vectors $\mathbf{V}_X^{\{k\}}$ of length $\bar{L}$.

To construct our EMs we introduce the operator $\mathrm{d}x_j$, which assigns to vector $\mathbf{V}$ its jth component, i.e. $\mathrm{d}x_j(\mathbf{V}) = V(j)$, and combines them in the wedge product defined as [10]

$$\bigwedge_{i=0}^{l-1} \mathrm{d}x_{j_i}(\mathbf{V}_0,\ldots,\mathbf{V}_{l-1}) \equiv \mathrm{Det}\left(\mathrm{d}x_{j_i}(\mathbf{V}_m)\right)_{i,m=0,\ldots,l-1}. \tag{9}$$

The wedge product is completely antisymmetric with respect to interchange of any two vectors in its argument, and is zero for any two repeated arguments.

Writing the ordered set of vectors obtained from the action of all $I_X^{\{k\}}$ operators at a given locus as $\{\mathbf{V}\} \equiv \left\{\mathbf{V}_0^{(k_1,\ldots,k_n)},\ldots,\mathbf{V}_{l-1}^{(k_1,\ldots,k_n)}\right\}$, we define the quantities

$$D_n^{(k_1,\ldots,k_n)} \equiv l^2\left\{\sum_{j_0<\cdots<j_{l-1}}\left|\bigwedge_{i=0}^{l-1}\mathrm{d}x_{j_i}(\{\mathbf{V}\})\right|^2\right\}^{2/l}. \tag{10}$$

These are the objects that we study in the rest of the paper, and as we show below, they are EMs.

For a given n there are $\binom{N}{n}$ quantities $D_n^{(\{k\})}$, except when $n = N/2$ with N even, in which case there are half this number since, there are then only $\frac{1}{2}\binom{N}{N/2}$ distinct bipartite divisions. For example, for four qubits we have four $D_1^{(k)}$ and three $D_2^{(k_1,k_2)}$ measures. These are not necessarily all independent. For $n = 1$, we recover the quantities of equation (5) introduced by MW. Furthermore, for the two-qubit system $|\Psi\rangle = \sum_{ij=0}^{1} A_{ij}|ij\rangle$, we have $D_1 = 4|\mathrm{Det}\,A|^2 = \mathcal{C}^2$, the square of the concurrence—itself an EM.

3. Properties

3.1. The quantities $D_n^{(\{k\})}$ are entanglement monotones

In the appendix, we investigate the properties of the quantities $D_n^{(\{k\})}$ under unitary transformations. We show that not only are they invariant under single qubit unitary

transformations but, writing the wavefunction as $|\Psi\rangle = \sum_{i=0}^{l-1} |\phi_i\rangle|V_i\rangle$ where $|\phi_i\rangle$ are states of the n reduced qubits, we show that $D_n^{(\{k\})}$ is also invariant under unitary transformations of the whole subspace spanned by $|\phi_i\rangle$. Consequently, writing $|\Psi\rangle$ in the Schmidt decomposition, $|\Psi\rangle = \sum_{i=0}^{l-1} |\phi_i\rangle|\tilde{\mathbf{V}}_i\rangle$ with $|\tilde{\mathbf{V}}_i\rangle$ orthogonal but not normalized, leaves $D_n^{(\{k\})}$ unaltered. The Schmidt coefficients are $\langle\tilde{\mathbf{V}}_i|\tilde{\mathbf{V}}_i\rangle \geqslant 0$.

Using this decomposition, we show in the appendix that $D_n^{(\{k\})}$ can be written as

$$D_n^{(\{k\})} = l^2 \left\{ \prod_{i=0}^{l-1} \langle\tilde{\mathbf{V}}_i|\tilde{\mathbf{V}}_i\rangle \right\}^{2/l}. \tag{11}$$

This form shows that $D_n^{(\{k\})}$ is indeed an EM. From Vidal [2], we know that all entanglement monotones of a bipartite system can be expressed as $g(\{\alpha_i\})$, where g is a symmetric, concave function of the Schmidt coefficients $\{\alpha_i\}$. Equation (11) shows $D_n^{(\{k\})}$ to be equal (up to normalization) to the geometric mean of the square of the Schmidt coefficients $\left(g = \sqrt[l]{\alpha_1^2 \ldots \alpha_l^2}\right)$. Since for $n > 1$ this is manifestly a concave function, it follows immediately that $D_{n>1}^{(\{k\})}$ is an EM. The quantities $D_{n=1}^{(k)}$ are also EMs, as can be seen by comparison with the linear entropy, below.

The power $2/l$ in equation (11) is chosen to be the maximum that ensures that $D_n^{(\{k\})}$ is an EM for all n. This choice is justified further in the appendix, where we show that with it, $D_n^{(\{k\})}$ transforms under local POVM in the same way as do the concurrence-squared and the tangle.

These quantities have the interesting geometric interpretation as being proportional to the square of the length of side of the hypercube with the same volume as the parallelogram defined by the set of vectors $\{\mathbf{V}\}$.

3.2. Comparison with linear entropies

Since the construction of $D_n^{(\{k\})}$ is predicated on a bipartite division of the system, we now compare $D_n^{(\{k\})}$ with a more familiar EM based on the same division, namely the linear entropy of qubits $\{k\}$ with the rest of the system. The linear entropies are defined as

$$S_n^{(k_1,\ldots,k_n)} \equiv \eta_n \left[1 - \mathrm{Tr}\left(\rho_{k_1,\ldots,k_n}^2\right)\right] \tag{12}$$

where $\rho_{k_1,\ldots,k_n}$ is the reduced density matrix of qubits $k_1, \ldots, k_n$, and $\eta_n = 2^n/(2^n - 1)$ provides suitable normalization [11].

By utilizing the Schmidt decomposition as above, we write

$$S_n^{(\{k\})} = \eta_n \left(1 - \sum_{i=0}^{l-1} \langle\tilde{\mathbf{V}}_i|\tilde{\mathbf{V}}_i\rangle^2\right). \tag{13}$$

Thus, $S_n^{(\{k\})}$ is constructed from the sum of the squares of the Schmidt coefficients, and is an EM since $1 - x^2$ is a concave function.

For $n = 1$, this relation, plus the normalization of the Schmidt coefficients, $\langle\mathbf{V}_0|\mathbf{V}_0\rangle + \langle\mathbf{V}_1|\mathbf{V}_1\rangle = 1$, shows that $D_1^{(k)} = S_1^{(k)}$. This was the relation noted by Brennen in connection with the original MW measure [8]. For $n \geqslant 2$ this identification does not hold.

3.3. Simple applications

Considering the form of $D_n^{(\{k\})}$ from equation (11), it is clear that the minimum value of $D_n^{(\{k\})}$ is zero, occurring when any $\langle\tilde{\mathbf{V}}_i|\tilde{\mathbf{V}}_i\rangle = 0$. The measure $D_n^{(\{k\})}$ is maximized when $\langle\tilde{\mathbf{V}}_i|\tilde{\mathbf{V}}_i\rangle = 1/l; \forall i$, from which we see that $D_n^{(\{k\})}$ is normalized such that $\max D_n^{(\{k\})} = 1$.

If the block of qubits $\{k\}$ is separable from the rest of the system, then $D_n^{(\{k\})} = S_n^{(\{k\})} = 0$. Moreover, if any of the qubits in $\{k\}$ separates, then $D_n^{(\{k\})} = 0$, whereas the corresponding entropy would be nonzero in general. Neither $S_n^{(\{k\})}$ nor $D_n^{(\{k\})}$ are necessarily zero if there are separable qubits in the conjugate locus $\{\bar{k}\}$.

For the N-qubit GHZ (Greenberger–Horne–Zeilinger [12]) state, $|\gamma\rangle \equiv 2^{-1/2}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$, one has $D_1^{(k)} = 1$ and $D_{n\geqslant 2}^{(\{k\})} = 0$. For the N-particle W-state, $|\omega\rangle \equiv N^{-1/2}\sum_{j=1}^{N}|0\rangle^{\otimes j-1}\otimes|1\rangle\otimes|0\rangle^{\otimes N-j}$, we find $D_1^{(k)} = 4(N-1)/N^2$ and $D_{n\geqslant 2}^{(\{k\})} = 0$. In contrast $S_{n\geqslant 2}^{(\{k\})}$ is nonzero for both these states; $S_n^{(\{k\})}(\gamma) = \eta_n/2$ and $S_n^{(\{k\})}(\omega) = 2\eta_n(N-n)n/N^2$ for all n [11].

A necessary, but not sufficient, requirement for at least one $D_n^{(\{k\})}$ to be nonzero is that $|\Psi\rangle$ must be a superposition of at least 2^n states. For example, the four-qubit state $1/2(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)$ has $D_2^{(1,2)} = D_2^{(1,4)} = 1$, $D_2^{(1,3)} = 0$.

Finally, to demonstrate that information is contained in $D_{n\geqslant 2}^{(\{k\})}$ that is not in $S_{n\geqslant 2}^{(\{k\})}$, consider the two four-qubit states

$$|\psi_\pm\rangle = \tfrac{1}{\sqrt{8}}(|0000\rangle + |0001\rangle \pm |0110\rangle \pm |0111\rangle \pm |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle). \quad (14)$$

Both states have the same linear entropies but, whereas $|\psi_+\rangle$ has all $D_2^{(\{k\})} = 0$, $|\psi_-\rangle$ has $D_2^{(1,2)} = D_2^{(1,3)} = 1/2$, $D_2^{(1,4)} = 0$. This shows $D_n^{(\{k\})}$ to be independent of $S_n^{(\{k\})}$ for $n \geqslant 2$.

After these simple examples, we now consider in detail the case of four qubits.

4. Four qubits

Consider the general four-qubit state

$$|\Psi\rangle = \sum_{X=0}^{15} a_X|X\rangle \quad (15)$$

in the usual decimal representation of a bit-string. The explicit forms of $S_1^{(\{k\})} = D_1^{(\{k\})}$ and $S_2^{(\{k\})}$ are unenlightening. However, the set of three $D_2^{(\{k\})}$ operators reveals the use of this construction. We find

$$D_2^{(1,2)} = 16\left|\mathrm{Det}\begin{pmatrix} a_0 & a_4 & a_8 & a_{12} \\ a_1 & a_5 & a_9 & a_{13} \\ a_2 & a_6 & a_{10} & a_{14} \\ a_3 & a_7 & a_{11} & a_{15} \end{pmatrix}\right| \quad (16)$$

$$D_2^{(1,3)} = 16\left|\mathrm{Det}\begin{pmatrix} a_0 & a_2 & a_8 & a_{10} \\ a_1 & a_3 & a_9 & a_{11} \\ a_4 & a_6 & a_{12} & a_{14} \\ a_5 & a_7 & a_{13} & a_{15} \end{pmatrix}\right| \quad (17)$$

$$D_2^{(1,4)} = 16\left|\mathrm{Det}\begin{pmatrix} a_0 & a_1 & a_8 & a_9 \\ a_2 & a_3 & a_{10} & a_{11} \\ a_4 & a_5 & a_{12} & a_{13} \\ a_6 & a_7 & a_{14} & a_{15} \end{pmatrix}\right|. \quad (18)$$

These are the moduli of the three fundamental algebraic invariants found by Luque *et al* for four qubits using classical invariant theory [3]. The status of these algebraic invariants is

elevated to EMs once the modulus is taken. Note that only two of these three quantities are independent.

Furthermore, our measures $D_n^{(\{k\})}$ are useful in distinguishing between types of entanglement in four-qubit systems. The states of N qubits may be grouped into families under the principle that all the members of a family may be converted into one another using LOCC with some finite, but not necessarily certain, probability of success. States connected in this way are said to be related by SLOCC, standing for *stochastic* LOCC [14, 9, 13]. In deciding which family an arbitrary state belongs to, we can consider the EMs. Since EMs are non-increasing under LOCC, the property of a state having an EM equal to zero is preserved under LOCC. These zero EMs may therefore serve to differentiate between the families.

Verstraete *et al* [9] have analysed the properties of four-qubit systems under SLOCC, and have demonstrated that there are nine distinct families of four-qubit states. The generic family of four qubits G_{abcd} is identified as being the only family with hyperdeterminant $\Delta \neq 0$ and this is the only family having genuine four-particle entanglement. Of the remaining eight families, we can distinguish three different groups based on the $D_2^{(\{k\})}$ measures. The families L_{abc_2} and L_{ab_3} have no zero $D_2^{(\{k\})}$, as indeed does the family G_{abcd}. Families $L_{a_2b_2}$ and L_{a_4} have a single zero $D_2^{(\{k\})}$, and the remaining four families have all $D_2^{(\{k\})} = 0$. This classification holds for all generic members of each family, but may fail in special cases of zero measure, such as the completely separable state that belongs to the generally non-separable family L_{abc_2}.

It is clear that further EMs are required to complete this classification. The linear entropies are not useful in this context, as they have no obvious relation to the SLOCC families, although they may be used to identify separable states.

In the context of SLOCC, we also mention the set of EMs introduced by Verstraete *et al*, both for the four qubit system [9], and more generally [13]. These are different to the quantities introduced here, but share the interesting similarity of being of the form of the modulus of the sum of products of the wavefunction amplitudes a_X (see equation (1)) combined with antisymmetric tensors. These quantities bear a closer relation to the hyperdeterminant than do the measures $D_n^{(\{k\})}$ [13].

5. Discussion

We have introduced the algebraic invariants $D_n^{(\{k\})}$, which are entanglement monotones for pure states of N qubits. They arise from considering all bipartitions of the system, and we have compared $D_n^{(\{k\})}$ with the linear entropies $S_n^{(\{k\})}$ of the same partitions.

There are, in principle, an infinite number of EMs based upon a given bipartite decomposition of an N-qubit state, as any concave, symmetric function of the Schmidt coefficients is an EM. The usefulness of the linear entropy as indicator of separability is clear, and the $D_n^{(\{k\})}$ may be seen as complementary to the linear entropies. Whereas $S_n^{(\{k\})}$ are based upon the sum of the squares of the Schmidt coefficients (a non-decreasing monotone), $D_n^{(\{k\})}$ are based on their geometric mean. Furthermore, $D_n^{(\{k\})}$ are able to reproduce the fundamental algebraic invariants, at least for the small numbers of qubits ($N \leqslant 4$) for which these quantities are known. It is hoped that $D_n^{(\{k\})}$ will be useful in the determination of AIs and the classification of systems with $N \geqslant 5$ qubits.

From the form of $D_n^{(\{k\})}$ given in equation (11), it is clear how to extend the definition to parties with Hilbert spaces of greater dimension, as the geometric mean construction is not dependent on this dimension being two. Finally we note that it is a simple step to introduce the average or minimum of the quantities $D_n^{(\{k\})}$ and obtain a single EM for a given n in the

same way as has been done for the linear entropies [11, 15]. The behaviour of such quantities is left for future work.

Acknowledgments

I am grateful to CWJ Beenakker for discussions and advice. This work was supported by the Dutch Science Foundation NWO/FOM.

Appendix A. Transformation properties of $D_n^{(\{k\})}$

In this appendix we show that $D_n^{(\{k\})}$ is invariant under local unitary transformations. We also consider the action of a POVM (positive operator-valued measurement) on the system.

A.1. Invariance under unitary transformations of reduced qubits

We first consider the invariance of $D_n^{(\{k\})}$ with respect to unitary transformations of a qubit belonging to $k_1, \ldots, k_n$, which we take to be the first without lack of generality. We write the wavefunction as

$$|\Psi\rangle = \sum_{X=0}^{l/2-1} (|0\rangle|\alpha_X\rangle|\mathbf{V}_{0,X}\rangle + |1\rangle|\alpha_X\rangle|\mathbf{V}_{1,X}\rangle) \tag{A1}$$

where $|\alpha_X\rangle$ are basis states of the other qubits in $\{k\}$, and $|\mathbf{V}_{(i=0,1),X}\rangle$ are the same vectors as before, except that we treat the index (i) belonging to the first qubit separately from the rest (X). We thus write the set of vectors $\{\mathbf{V}\}$ as $\{\mathbf{V}_{0,X}, \mathbf{V}_{1,X}\}$, with the vectors $\mathbf{V}_{0,X}$ to the left of $\mathbf{V}_{1,X}$. We act on the first qubit with a general unitary operator U, giving the wavefunction

$$U|\Psi\rangle = \sum_{X=0}^{l/2-1} |0\rangle|\alpha_X\rangle(U_{00}|\mathbf{V}_{0,X}\rangle + U_{10}|\mathbf{V}_{1,X}\rangle) + |1\rangle|\alpha_X\rangle(U_{01}|\mathbf{V}_{0,X}\rangle + U_{11}|\mathbf{V}_{1,X}\rangle). \tag{A2}$$

Define

$$F_{\{j\}}(|\Psi\rangle) \equiv \bigwedge_{i=0}^{l-1} \mathrm{d}x_{j_i}(\{\mathbf{V}_{0,X}, \mathbf{V}_{1,X}\}) \tag{A3}$$

in terms of which

$$D_n^{(\{k\})} = l^2 \left\{ \sum_{j_0<\cdots<j_{l-1}} F_{\{j\}}(|\Psi\rangle) F_{\{j\}}(|\Psi^*\rangle) \right\}^{2/l}. \tag{A4}$$

From equation (A2), the wedge product for the transformed wavefunction is

$$F_{\{j\}}(U|\Psi\rangle) = \bigwedge_{i=0}^{l-1} \mathrm{d}x_{j_i}(\{U_{00}\mathbf{V}_{0,X} + U_{10}\mathbf{V}_{1,X}\}, \{U_{01}\mathbf{V}_{0,X} + U_{11}\mathbf{V}_{1,X}\}). \tag{A5}$$

Since $\bigwedge \mathrm{d}x$ is linear, and zero when any two of its arguments are the same, we can write

$$F_{\{j\}}(U|\Psi\rangle) = \sum_{k_0,\ldots,k_{l-1}} \bigwedge_{i=0}^{l-1} \mathrm{d}x_{j_i}\left(\left\{U_{k_X 0}\mathbf{V}_{k_X,X}\right\}, \left\{U_{\bar{k}_X 1}\mathbf{V}_{\bar{k}_X,X}\right\}\right) \tag{A6}$$

where $k_X = 0$, 1 and $\bar{k}_X = (1+k_X) \bmod 2$. We proceed by rearranging the vectors in the above expression such that all $\mathbf{V}_{0,X}$ stand to the left of $\mathbf{V}_{1,X}$, and collecting the appropriate elements

of U with signs given by the antisymmetry of the wedge product. The term that requires no interchange of vectors acquires a forefactor $U_{00}^{l/2}U_{01}^{l/2}$, and there is only one such term. There are $l/2$ terms that require a single exchange of vectors. These terms have a forefactor $U_{00}^{l/2-1}U_{10}U_{01}^{l/2-1}U_{11}$, and acquire a minus sign due to the antisymmetry. Proceeding similarly for all the terms we arrive at

$$\begin{aligned} F_{\{j\}}(U|\Psi\rangle) &= \sum_{k=0}^{l/2}\binom{l/2}{k}(U_{00}U_{11})^{l/2-k}(-U_{01}U_{10})^{k}\bigwedge_{i=0}^{l-1}\mathrm{d}x_{j_i}(\{\mathbf{V}_{0,X}\},\{\mathbf{V}_{1,X}\}) \\ &= (\mathrm{Det}U)^{l/2}\,F_{\{j\}}(|\Psi\rangle) \end{aligned} \tag{A7}$$

Therefore, the effect of a unitary transformation on any of the reduced qubits is to multiply each of the terms in $D_n^{(\{k\})}$ by $|\mathrm{Det}\,U|^2 = 1$. Thus $D_n^{(\{k\})}$ is invariant under such transformations.

This invariance also holds when we consider general transformations of the entire n-qubit subsystem defined by the locus of $D_n^{(\{k\})}$. The operation of the most general $l \times l$ unitary operator on this n-qubit Hilbert space multiplies each term in $D_n^{(\{k\})}$ by $|\mathrm{Det}\,U|^{4/l} = 1$, demonstrating the invariance as above. Such invariance is not a requirement for being an EM, but it will be of use in the following.

A.2. Invariance under unitary transformation of wedge-product

The unitary invariance of the system under transformations of the entire reduced qubit Hilbert space enables us to write the state vector in a Schmidt decomposition $|\Psi\rangle = \sum_{i=0}^{l-1}|\phi_i\rangle|\tilde{\mathbf{V}}_i\rangle$ without altering $D_n^{(\{k\})}$. The vectors $|\tilde{\mathbf{V}}_i\rangle$ are orthogonal but not normalized. To demonstrate that $D_n^{(\{k\})}$ is invariant under local unitary transformations of the qubits inside the wedge product, we begin by writing $D_n^{(\{k\})}$ as

$$D_n^{(\{k\})} = l^2\left\{\sum_{j_0<\cdots<j_{l-1}}\bigwedge_{i=0}^{l-1}\mathrm{d}x_{j_i}(\{\tilde{\mathbf{V}}\})\bigwedge_{i=0}^{l-1}\mathrm{d}x_{j_i}(\{\tilde{\mathbf{V}}^*\})\right\}^{2/l}. \tag{A8}$$

We change the sum to include all values $\{j_i\}$, and write the wedge-products as tensors

$$D_n^{(\{k\})} = \frac{l^2}{l!}\left\{\sum_{j_0,\ldots,j_{l-1}}\left(\bigwedge_{i=0}^{l-1}\mathrm{d}x(\{\tilde{\mathbf{V}}\})\right)_{\{j_i\}}\left(\bigwedge_{i=0}^{l-1}\mathrm{d}x\left(\{\tilde{\mathbf{V}}^*\}\right)\right)_{\{j_i\}}\right\}^{2/l}. \tag{A9}$$

The tensor $\bigwedge_{i=0}^{l-1}\mathrm{d}x(\{\tilde{\mathbf{V}}\})$ corresponds to a sum of ordered l-tuples of vectors

$$\bigwedge_{i=0}^{l-1}\mathrm{d}x(\{\tilde{\mathbf{V}}\}) = \sum_{\{k\}}\epsilon_{\{k\}}\left[\tilde{\mathbf{V}}_{k_0},\ldots,\tilde{\mathbf{V}}_{k_{l-1}}\right] \tag{A10}$$

with antisymmetric coefficients $\epsilon_{\{k\}} = \pm 1$ from the determinant of equation (9). Using this form, we have

$$D_n^{(\{k\})} = \frac{l^2}{l!}\left\{\sum_{j_0,\ldots,j_{l-1}}\left(\sum_{\{k\}}\epsilon_{\{k\}}\left[\tilde{\mathbf{V}}_{k_0},\ldots,\tilde{\mathbf{V}}_{k_{l-1}}\right]\right)_{\{j_i\}}\left(\sum_{\{k'\}}\epsilon_{\{k'\}}\left[\tilde{\mathbf{V}}^*_{k'_0},\ldots,\tilde{\mathbf{V}}^*_{k'_{l-1}}\right]\right)_{\{j_i\}}\right\}^{2/l}. \tag{A11}$$

Summing over the j indices, and recognizing the scalar product of two vectors, we have

$$D_n^{(\{k\})} = \frac{l^2}{l!}\left\{\sum_{\{k\},\{k'\}}\epsilon_{\{k\}}\epsilon_{\{k'\}}\langle\tilde{\mathbf{V}}_{k_0}|\tilde{\mathbf{V}}_{k'_0}\rangle\langle\tilde{\mathbf{V}}_{k_1}|\tilde{\mathbf{V}}_{k'_1}\rangle\cdots\langle\tilde{\mathbf{V}}_{k_{l-1}}|\tilde{\mathbf{V}}_{k'_{l-1}}\rangle\right\}^{2/l}. \tag{A12}$$

We now use the orthogonality of $|\tilde{\mathbf{V}}\rangle$ from the Schmidt decomposition, and the fact that $\epsilon_{\{k\}}^2 = 1$ to write

$$D_n^{(\{k\})} = l^2 \left\{ \prod_{i=0}^{l-1} \langle \tilde{\mathbf{V}}_i | \tilde{\mathbf{V}}_i \rangle \right\}^{2/l}. \tag{A13}$$

Thus we see that $D_n^{(\{k\})}$ is the product of Schmidt coefficients, and thus invariant with respect to unitary transformations of the qubits inside the wedge-product.

A.3. Action of POVM

That $D_n^{(\{k\})}$ is an EM follows from the above results and the argument given in the main text. Here we give an alternative demonstration, based on that used by Dür *et al* in establishing the tangle as an EM [14], which provides justification for the power $2/l$ chosen in the definition of $D_n^{(\{k\})}$.

Any local protocol can be decomposed into POVMs acting on a single qubit. As any POVM can be further decomposed into a sequence of two-outcome POVMs, we need only to demonstrate the non-increasing of $D_n^{(\{k\})}$ under the action of a two-outcome POVM to show that it is an EM.

Let the two elements of the POVM be A_1 and A_2 such that $A_1^\dagger A_1 + A_2^\dagger A_2 = 1$. Using singular-value decompositions for these matrices, we have $A_i = U_i X_i V$, where U_i, V are unitary, V is the same for both elements and $X_{1,2}$ are the diagonal matrices (a, b) and $(\sqrt{1-a^2}, \sqrt{1-b^2})$.

Consider the initial state $|\psi\rangle$, which possesses the measure $D_n^{(\{k\})}(\psi)$. We write the (unnormalized) states obtained by the action of the POVM on the state as $|\widetilde{\phi}_i\rangle = A_i|\psi\rangle$. Normalizing them, we have $|\phi_i\rangle = |\widetilde{\phi}_i\rangle/\sqrt{p_i}$, where $p_i \equiv \langle\widetilde{\phi}_i|\widetilde{\phi}_i\rangle$ with $p_1 + p_2 = 1$.

In analogy with the tangle, we wish to show that $(D_n^{(\{k\})})^\nu$, $0 < \nu \leqslant 1$ is non-increasing, on average, under the action of the POVM, i.e.

$$\left\langle \left(D_n^{(\{k\})}\right)^\nu \right\rangle \leqslant \left(D_n^{(\{k\})}\right)^\nu (\psi) \tag{A14}$$

with

$$\left\langle \left(D_n^{(\{k\})}\right)^\nu \right\rangle = p_1 \left(D_n^{(\{k\})}\right)^\nu (\phi_1) + p_2 \left(D_n^{(\{k\})}\right)^\nu (\phi_2) \tag{A15}$$

for all possible choices of POVM and states $|\psi\rangle$. Since $D_n^{(\{k\})}$ is invariant under unitary transformations, we may omit the matrices U_i from the decomposition of the POVM, such that $D_n^{(\{k\})}(\phi_i) = D_n^{(\{k\})}(X_i V\psi/\sqrt{p_i})$.

Performing the POVM on one of the reduced qubits, we evaluate $\left\langle \left(D_n^{(\{k\})}\right)^\nu \right\rangle$ to be

$$\left\langle \left(D_n^{(\{k\})}\right)^\nu \right\rangle = \left\{ p_1 \left(\frac{a^2 b^2}{p_1^2} \right)^\nu + p_2 \left(\frac{(1-a^2)(1-b^2)}{p_2} \right)^\nu \right\} \left(D_n^{(\{k\})}\right)^\nu (\psi). \tag{A16}$$

This is exactly the same dependence as found by Dür *et al* for the tangle [14]. For all $0 < \nu \leqslant 1$ the forefactor in equation (A16) is not greater than one and thus $\left\langle \left(D_n^{(\{k\})}\right)^\nu \right\rangle \leqslant \left(D_n^{(\{k\})}\right)^\nu (\psi)$, reiterating the conclusion that $D_n^{(\{k\})}$ is an EM. This result shows that the choice of the power $2/l$ in the definition of $D_n^{(\{k\})}$ is the natural choice, as it makes $D_n^{(\{k\})}$ transform under local POVMs in the same way as do the concurrence-squared and the tangle.

References

[1] Linden N, Popescu S and Sudbery A 1999 *Phys. Rev. Lett.* **83** 243
Carteret H A, Linden N, Popescu S and Sudbery A 1999 *Found. Phys.* **29** 527
[2] Vidal G 2000 *J. Mod. Opt.* **47** 355
[3] Luque J-G and Thibon J-Y 2003 *Phys. Rev.* A **67** 042303
Briand E, Luque J-G and Thibon J-Y 2003 *J. Phys. A: Math. Gen.* **36** 9915
[4] Miyake A 2003 *Phys. Rev.* A **67** 012108
[5] Wootters W K 1998 *Phys. Rev. Lett.* **80** 2245
[6] Coffman V, Kundu J and Wootters W K 2000 *Phys. Rev.* A **61** 052306
[7] Meyer D A and Wallach N R 2002 *J. Math. Phys.* **43** 4273
[8] Brennen G K 2003 *Quantum Inf. Comput.* **3** 619
[9] Verstraete F, Dehaene J, De Moor B and Verschelde H 2002 *Phys. Rev.* A **65** 052112
[10] Zwillinger D (ed) 1996 *Standard Mathematical Tables and Formulae* (Boca Raton, FL: CRC Press)
[11] Scott A J 2003 *Preprint* quant-ph/0310137
[12] Greenberger D M, Horne M and Zeilinger A 1989 *Bell's Theorem, Quantum Theory and Conceptions of the Universe* ed M Kafatos (Dordrecht: Kluwer)
[13] Verstraete F, Dehaene J and De Moor B 2003 *Phys. Rev.* A **68** 012103
[14] Dür W, Vidal G and Cirac J I 2000 *Phys. Rev.* A **62** 062314
[15] Emary C and Beenakker C W J 2004 *Phys. Rev.* A **69** 032317